***Linear Algebra***

1. A New Isomorphism: The Matrix
   1. Given a system of linear equations, we can arrange the equations into a matrix based on the variable (or power of the variable) corresponding to each coefficient.
      1. EX: 3x+ y = 7 ⇒



9x-8y = 8

* 1. Basics
     1. An *n* x *m* matrix contains *n* rows and *m* columns whose elements are as follows:



* + 1. *Coefficient Matrix-* A matrix containing only the coefficients of the variables in a system of equations.
    2. *Augmented Matrix-* A matrix containing both the coefficients and the constant terms in a system of equations (see EX Iai)
    3. *Square Matrix*- A matrix where the number of rows equals the number of columns (*n* x *n*).
    4. *Diagonal Matrix-* A matrix wherein all elements above and below the main diagonal are zero.
       1. *Main Diagonal-* The diagonal from the top left element to the lower right one.
    5. *Upper/Lower Triangular Matrix-* A matrix wherein all elements above/below the main diagonal are zero.
  1. Column Vectors
     1. *Column Vector-* A matrix with only one column, sometimes denoted as ‘vector’ only;
        1. *Components-* The entries in a vector (column or row).
        2. *Standard Representation of Vectors-*



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is normally represented in the Cartesian plane by a directed line segment from the origin to the point (x,y). Vectors are traditionally allowed to slide at will having no fixed position, only direction and magnitude.

1. Reduced Row Echelon Form (RREF).
   1. Matrices can be manipulated with elementary row operations without compromising the isomorphism (losing answers).
      1. Elementary Row Operations:
         1. Multiplication by a nonzero scalar (real number)
         2. Adding one row to another
         3. Swapping Rows
   2. *Reduced Row Echelon Form (RREF)-* A matrix is in RREF⇔
      1. The leftmost non-zero entry in every non-zero row is a one.
      2. Every entry in the column containing a leading one is zero.
      3. Every row below a row containing a leading one has a leading one to the right.
   3. *Rank-* The number of leading 1s in a matrix’s RREF is the rank of that matrix; Consider A, an *n* x *m* matrix.
      1. If *rank* (A) = m, then there exists only one solution to the system.
      2. If *rank* (A) < m, then the system has either infinitely many or no solutions.
   4. This process of reduction carries three distinct possibilities:
      1. The RREF of the coefficient matrix is the ‘identity matrix’ (rows containing only zeroes are admissible as long as the remainder represents an identity matrix). Then, there exists only one solution to the system of equations, and reintroducing the variables will give it (multiply by the column vector containing the variables in their respective order).
         1. *Identity Matrix-* A matrix containing 1s on the main diagonal and 0s elsewhere.
         2. This is only possible when there are at least as many equations as unknowns.
      2. The matrix reduction produces a contradiction of the type ‘0 = C’ Э C ∈ **R.**
      3. The matrix reduction fails to produce a RREF conforming to i or a contradiction. This occurs when there is a variable in the system that is not dependent on the others and therefore multiple correct solutions exist.
         1. *Free variable*- A variable in a system which is not dependent on any of the others and therefore does not reduce out of the matrix.
         2. To express this solution, reintroduce variables and simply solve for the dependent (leading) variables. The free variable(s) is set equal to itself.
         3. Example:



⇒ =

* + - 1. It may be helpful to think of the last column as separated from the rest of the matrix by an ‘=’.
  1. Geometry
     1. Matrices have strong ties to geometrical concepts and the insight necessary to solve many linear algebra problems will be found by considering the geometric implications of a system (and corresponding matrices).
     2. Considering the above example, it’s clear our system represents three planes (three variables in each equation). Thus it should not surprise us that the intersections of three planes (the solutions) can either not occur (parallel planes, ii above) or take the form of a point (i above), a line (one free variable), or plane (two free variables).

1. Matrix Algebra
   1. Matrix Addition
      1. Matrix addition is accomplished by simply adding elements in the same position to form a new matrix.
   2. Scalar Multiplication
      1. The scalar is multiplied by every element individually.
   3. Matrix-Vector Multiplication
      1. If the number of rows in the column vector matches the number of columns in the matrix:



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Otherwise the solution is undefined.

* + 1. This is often defined in terms of the columns or rows of A (it’s not hard to translate)
    2. It’s helpful to think off placing the column vector horizontally above the matrix, multiplying downward, then summing for each row.
  1. Algebraic Rules
     1. If A is an *n* x *m;* and x and y are vectors in **R**m; and k is a scalar, then



1. Linear Transformations
   1. Matrix Form of a Linear System
      1. A linear system can be written in matrix form as where A is the ‘matrix of transformation’, is the column vector containing the variables of the system, and is the constant terms to which the variables are equal.



* 1. *Linear Transformation*
     1. A function T from Rm to Rn for which there exists an *n* x *m* matrix AЭ for all in Rm and



* + - 1. for all and ∈ Rm



* + - 1. for all for all ∈ Rm and all scalars k



* 1. Finding the ‘Matrix of Transformation’, A
     1. *Standard Vectors-* The vectors in Rm that contain a 1 in the position noted in their subscript and 0s in all others.



* + 1. Using the standard vectors,



Э represents the ith column in A.



* + 1. *Identity Transformation*
       1. The transformation that returns unchanged and thus has the identity matrix as its matrix of transformation.



* 1. Geometry
     1. Linear transformations can be found for many geometrical operations such as rotation, scaling, projection, translation, etc.

1. Composing Transformations and Matrix Multiplication
   1. Just as we can compose functions and generate another function, so can we compose linear transformations and generate another linear transformation. This composing can be represented as .



* 1. To translate this into a new linear transformation, we need to find the new matrix of transformation C=BA; this process is known as ‘matrix multiplication’.
     1. Matrix Multiplication
        1. Let B be an *n* x *p* matrix and A a *q* x *m* matrix. The product BA is defined ⇔ p = q
        2. If B is an *n* x *p* matrix and A a *q* x *m* matrix, then the product BA is defined as the linear transformation for all in Rm. The product BA is an *n* x *m* matrix.



* + - 1. Arrange the two matrices as follows (the order is important!):



For each new element in the matrix, you must multiply the elements of the old matrices along the two lines that cross at its position, then sum the products. In this case, the new element will be equal to a11b11+a21b12+a31b13. Repeat.

* + 1. Properties of Matrix Multiplication
       1. MATRIX MULTIPLICATION IS NONCOMMUTATIVE
          1. BA ≠ AB
          2. That means that what side of a matrix you write another matrix on matters! This is not normal so pay close attention!
          3. BA and AB both exist ⇔ A and B are square matrices
       2. Matrix Multiplication is associative
          1. (AB)C = A(BC)
       3. Distributive Property
          1. A(C+D) = AC + AD
          2. (A+B)C = AC + BC

Be careful! Column-Row rule for matrix multiplication still applies!

* + - 1. Scalars
         1. (kA)B = A(kB) = k(AB)

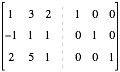
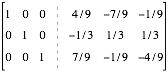
1. The Inverse of a Linear Transformation
   1. A Linear Transformation is invertible if its RREF is an identity matrix and therefore provides a single solution for any (ensures bijectivity)



* + 1. *Invertible Matrix-* A matrix which, when used as the matrix of transformation in a linear transformation, produces an invertible linear transformation.
    2. As stated earlier, this requires that the matrix in question to either be square, or for excess rows to 0 out.
    3. Additional Properties of an Invertible Matrix
       1. rref(A) = In
       2. rank (A) = n
       3. im (A) = Rn
       4. ker (A) = { }



* + - 1. Column Vectors form a basis of Rn
      2. det (A) ≠ 0
      3. 0 fails to be an eigenvalue of A
  1. Finding the Inverse of a Matrix
     1. To find the inverse of a matrix A, combine A with the same-sized identity matrix as shown, then row-reduce A. When A is the identity matrix, the identity matrix will be A-1.



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* + 1. AA-1 = IA  and A-1A = IA
    2. A = ⇒ = A-1



* + 1. (AB)-1 = B-1A-1
  1. Geometry
     1. The invertibility or non-invertibility of a given matrix can also be viewed from a geometric perspective based on the conservation of information with the corresponding linear transformation. If any information is lost during the transformation, the matrix will not be invertible (and the contra positive).
     2. Consider two geometric processes, translation and projection. After a moment of thought, it should be obvious that, given knowledge of the translation, we could undo any translation we’re given. Conversely, given knowledge of the type of projection and the projection itself, there are still infinitely many vectors which could correspond to any vector on our plane. Thus, translation is invertible, and projection is not.

1. The Image and Kernel of a Linear Transformation
   1. *Linear Combinations*-A vector  in **R**n is called a linear combination of the vectors  in **R**n if there exists scalars x1, … , xm such that 
   2. *Span-* The set of all linear combinations of the vectors is called their span:



* 1. *Spanning Set-* A set of vectors ∈ V which can express ∀ ∈ V as a linear combination of themselves.



* + 1. span() = V



* 1. *Subspace of Rn*- A subset W of the vector space Rn is called a (linear) subspace of Rn if it has the following properties:
     1. W contains the zero vector in Rn
     2. W is closed under (vector) addition
     3. W is closed under scalar multiplication.
        1. ii and iii together mean that W is closed under linear combination.
  2. *Image of a Linear Transformation-* The image of the linear transformation is the span of the column vectors of A.



* + 1. im(T)= im(A) = where are the column vectors of A.



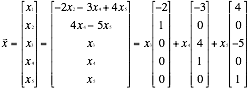
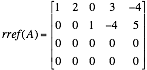
* + 1. The image of T: Rm → Rn is a subspace of the target space Rn or im(A) ⊆ Rn
       1. Properties
          1. The zero vector in Rn is in the image of T
          2. The image of T is closed under addition.
          3. The image of T is closed under scalar multiplication
  1. *Kernel of a Linear Transformation-* All zeroes of the linear function, i.e. all solutions to ; denoted ker(T) and ker (A).



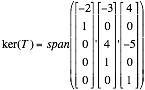
* + 1. The kernel of T: Rm → Rn is a subspace of the domain Rm or ker(A) ⊆ Rn
       1. Properties
          1. The zero vector in Rm is in the kernel of T
          2. The kernel is closed under addition
          3. The kernel is closed under scalar multiplication
    2. Finding the Kernel
       1. To find the kernel, simply solve the system of equations denoted by . In this case, since all the constant terms are zero, we can ignore them (they won’t change) and just find the rref (A).



* + - 1. Solve for the leading (dependent) variables
      2. The ker(A) is equal to the span of the vectors with the variables removed or just the system solved for the leading variables.



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1. Bases and Linear Independence
   1. *Redundant Vectors-* We say that a vector in the list is redundant if is a linear combination of the preceding vectors



* 1. *Linear Independence-* The vectors are called linearly independent if none of them is redundant; otherwise, they are called linearly dependent.



* + 1. ker



* + 1. rank



* 1. *Basis-* The vectors form a basis of a subspace V of Rn if they span V and are linearly independent (the vectors are required to be in V).



* 1. Finding a Basis
     1. To construct a basis, say of the image of a matrix A, list all the column vectors of A and omit the redundant vectors.
     2. Finding Redundant Vectors
        1. The easiest way to this is by ‘inspection’ or looking at the vectors (specifically 0 components) and noticing that if a vector has a zero it cannot produce anything in that position other than a 0.
        2. When this isn’t possible, we can use a subtle connection between the kernel and linear independence. The vectors in the kernel of a matrix correspond to linear relations in which the vectors are set equal to zero. Thus, solving for the free variable, we obtain a linear combination. Long story short, any column in a rref that doesn’t contain only one 1 and all else 0s, is redundant. Moreover, the values in that column of the matrix correspond to the scalars by which the other columns need be multiplied to produce the particular vector (note that the column won’t contain a scalar for itself).
     3. The number of vectors in a basis is independent of the Basis itself (all bases for the same subspace have the same number of vectors )
     4. The matrix representing a basis will always be invertible.

1. Dimensions
   1. *Dimension-* The number of vectors needed to form a basis of the subspace V, denoted dim (V)
      1. If dim (V) = m
         1. There exists at most m linearly independent vectors in V
         2. We need at least m vectors to span V
         3. If m vectors in V are linearly independent, then they form a basis of V
         4. If m vectors in V span V, then they form a basis of V
      2. *The Rank-Nullity Theorem:* For an *n* x *m* matrix A, m = dim (ima(A)) + dim(ker(A))
2. Coordinates
   1. Using the idea of basis and spanning set, we can create a new coordinate system for a particular subspace. This system records the constant terms needed to generate a particular vector in the subspace.
   2. Consider a basis B = ( ) of a subspace V of Rn. Then any vector, , in V can denoted uniquely by:



Where c1, … , cm represents the scalars needed to form a linear combination representing . c1, … , cm are called the *B-Coordinates* of .



* 1. Linearity of Coordinates
     1. If B is a basis of a subspace V of Rn, then



* 1. B-Matrix
     1. *B-Matrix-* The matrix that transforms into for a given Linear Transformation, T.



* + 1. Finding the B-Matrix
       1. B = where are the vectors in the Basis B (this is what we did with the standard vectors!)



* + - 1. B = S-1AS where S is the ‘Standard Matrix’ and A is the matrix of transformation for T.
         1. *Standard Matrix-* The matrix containing all the members of the spanning set.



* + - * 1. Whenever this relation holds between two *n* x *n* matrices A and B, we say that A and B are similar, i.e. the represent the same linear transformation with respect to different bases.

Similarity is an Equivalence Relation

1. Linear/Vector Spaces
   1. *Linear/Vector Space-* A set endowed with a rule for addition and a rule for scalar multiplication such that the following are satisfied:
      1. (f+g)+h = f+(g+h)
      2. f+g = g+f
      3. There exists a unique neutral element n in V such that f+n = f
      4. For each f in V, there exists a unique g such that f + g = 0
      5. k(f+g) = kf+kg
      6. (c+k)f = cf+kf
      7. c(kf) = (ck)f
      8. 1(f) = f
   2. Vector/Linear Spaces are often not traditional Rn! Yet, all (unless specified otherwise) terms/relations transfer wholesale.
      1. Example: Polynomials! Derivation! Integration!
         1. Hint: Pn means all polynomials of degree less than or equal to n
   3. Remember the definition of subspaces!
   4. Finding the Basis of a Linear Space (V)
      1. Write down a typical element of the space in general form (using variables)
      2. Using the arbitrary constants as coefficients, express your typical element as a linear combination of some (particular) elements of V.
         1. Make sure you’ve captured any relationships between the arbitrary constants!
         2. EX: In P2, the typical basis is [1, x, x2] from which any element of P2 can be constructed as a linear combination.
      3. Verify the (particular) elements of V in this linear combination are linearly independent; then they will form a basis of V.
2. Linear Transformations and Isomorphisms
   1. *Linear Transformation (Vector Space)-*A function T from a linear space V to a linear space W that satisfies:
      1. T(f + g) = T(f) + T(g)
      2. T(kf) = kT(f)

for all elements f and g of V and for all scalars k.

* 1. If T is finite dimensional, the rank-nullity theorem holds (analogous definitions of rank and nullity to earlier).
  2. Isomorphisms and Isomorphic Spaces
     1. *Isomorphism-* An invertible linear transformation.
     2. *Isomorphic Spaces-* Two linear/vector spaces are isomorphic iff there exists an isomorphism between them, symbolized by ‘≈’.
     3. Properties
        1. A linear transformation is an isomorphism ⇔ ker(T) = {0} and im(T) = W

Assuming our linear spaces are finite dimensional:

* + - 1. If V is isomorphic to W, then dim(V) = dim (W)
  1. Proving Isomorphic
     1. Necessary Conditions
        1. dim(V) = dim (W)
        2. ker(T) = {0}
        3. im(T) = W
     2. Sufficient Conditions
        1. 1 & 2
        2. 1 & 3
        3. T is invertible (you can write a formula)

1. The Matrix of a Linear Transformation
   1. *B-Matrix (B)*- The matrix which converts elements from the original space V expressed in terms of the basis B into the corresponding element of W, also expressed in terms of the basis of B.
      1. [ f ]B -----B----> [T( f )]B



* 1. *Change of Basis Matrix-* An invertible matrix which converts from a basis B to another basis U in the same vector space or [ f ]U = S [ f ]B where S or SB ⇒ U denotes the change of basis matrix.



* 1. As earlier, the equalities:
     1. AS = SB
     2. A = SBS-1
     3. B = S-1AS

hold for linear transformation (A is the matrix of transformation in the standard basis).

1. Orthogonality
   1. *Orthogonal-* Two vectors in Rn are orthogonal ⇔ ,perpendicular



* 1. *Length (magnitude or norm) of a vector*- , a scalar



* 1. *Unit Vector-* a vector whose length is 1, usually denoted



* + 1. A unit vector can be created from any vector by



* 1. *Orthonormal Vectors-* A set of vectors Э∀ vectors are both unit vectors and mutually orthogonal.
     1. *⇒* for any ,



* + 1. The above may come in handy for proofs, especially when combined with distributing the dot product between a set of orthonormal vectors and another vector.
    2. Properties
       1. Linearly Independent
       2. n orthonormal vectors form a basis for Rn
  1. Orthogonal Projection
     1. Any vector in Rn can be uniquely expressed in terms of a subspace V of Rn by a vector in V and a vector perpendicular to V (creates a triangle)



* + 1. Finding the Orthogonal Projection
       1. If V is a subspace of Rn with an orthonormal basis , then



* + - 1. This can be checked with



* 1. *Orthogonal Complement-* Given a subset V of Rn, the orthogonal complement, , consists of all vectors in Rn that are orthogonal to all vectors in V.



* + 1. This is equivalent to finding the kernel of the orthogonal projection onto V.
    2. Properties:
       1. is a subspace of Rn



* + 1. Given a span of V, you can find this by finding the kernel of the span turned on its side (i.e. the vector that is perpendicular to both spanning vectors)!
  1. Angle between Two Vectors



* + - 1. Cauchy-Schwarz Inequality ensures that this value is defined.
  1. Cauchy-Schwarz Inequality



* 1. *Gram-Schmidt Process*- an algorithm for producing an orthonormal basis from any basis.
     1. For the first vector, simply divide it by its length to create a unit vector.
     2. To find the next basis vector, first find



* + - 1. This becomes in the general case.



* + 1. Then,



* + 1. This procedure is simply repeated for every vector in the original basis.
       1. Keep in mind that, to simply the calculation, any can multiplied by a scalar (becomes a unit vector, so won’t effect the end result just the difficulty of the calculation).



1. Orthogonal Transformations and Orthogonal Matrices
   1. *Orthogonal Transformation-* A transformation, T, from Rn to Rn that preserves the length of vectors.
      1. Orthogonal transformations preserve orthogonality and angles in general (Pythagorean theorem proof).
      2. Useful Relations
         1. If T:Rn⇒Rn is orthogonal and , then



* 1. *Orthogonal Matrices-* The transformation matrix of an orthogonal transformation.
     1. Properties:
        1. The product, AB, of two orthogonal *n x n* matrices A and B is orthogonal
        2. The inverse A-1 of an orthogonal *n x n* matrix A is orthogonal
        3. A matrix is orthogonal ⇔ ATA=In or, equivalently, A-1=AT.
        4. The columns of an orthogonal matrix form an orthonormal basis of Rn.
  2. The Transpose of a Matrix
     1. The matrix created by taking the columns of the original matrix and making them the rows of a new matrix (and therefore the rows become columns).
        1. More officially, the transpose AT of A is the *n x m* matrix whose ijth entry is the jith entry of A.
        2. *Symmetric-* A square matrix A Э AT=A
        3. *Skew Symmetric-* A square matrix A Э AT= -A
        4. (SA)T=ATST
     2. If are two (column) vectors in Rn, then



* + - 1. This WILL come in handy
  1. The Matrix of an Orthogonal Projection
     1. Considering a subspace V of Rn with orthonormal basis , the matrix of orthogonal projection onto V is QQT Э



1. Inner Product Spaces
   1. *Inner Product-* a linear space V is a rule that assigns a real scalar (denoted by <f, g>) to any pair f, g of elements in V Э the following properties hold for all f, g, h in V, and all c in R:
      1. <f, g> = <g, f> (symmetry)
      2. <f+h, g> = <f, g> + <h, g>
      3. <cf, g> = c<f, g>
      4. <f, f> > 0, for all nonzero f in V (positive definiteness)
         1. This is the tricky one! It will often require, for matrices, that the kernel = {0} or the matrix is invertible.
   2. *Inner Product Space-* A linear space endowed with an inner product.
   3. *Norm-* The magnitude of an element f of an inner product space:



* 1. *Orthogonality-* Two elements, f and g, of an inner product space are called orthogonal (or perpendicular) if



* 1. *Distance-* if f and g are two elements of an inner product space,



* 1. Orthogonal Projection
     1. Analogous to a true vector space; if g1,…,gm is an orthonormal basis of a subspace of W of an inner product space V, then for all f in V.



1. Determinants
   1. *Determinant-* In general, a formula for calculating a value which summarizes certain properties of a matrix, namely invertible if ≠ 0.
      1. Properties:
         1. det(AT) = det(A)
         2. det (AB) = det(A)det(B)
   2. The Determinant of a *2 x 2* matrix



* 1. The Determinant of a *3 x 3* matrix and Beyond
     1. Geometrically, our matrix will be invertible ⇔ its column vectors are linearly independent (and therefore span R3). This only occurs if .



* + 1. We can find an equivalent value with the following formulas: or



* + - 1. These formulas are visibly represented by first picking a column (or row), then multiplying every element in that column (or row) by the determinant of the matrix generated by crossing out the row and column containing that element. The sign in front of each product alternates (and starts with positive).
      2. In this way, if we select the 1st column, our determinant will be: det (A) = a11det(A11) – a21det(A21) + a31det(A31)
         1. Aij represents the *2 x 2* matrix generated by crossing out the ith row and jth column of our *3 x 3* matrix.
      3. This definition is recursive! It allows us to find the determinant of a square matrix of any size by slowing reducing it until we reach the *2 x 2* case.
      4. Pick your starting row/column with care! 0s are your friends!
  1. The Determinant and Elementary Row Operations
     1. If the proceeding seemed a little daunting for large matrices, there exists a simple relationship between the elementary row operations and the determinant that will allow us to greatly increase the number of zeroes in any given matrix.
     2. Gauss-Jordan Elimination and Ties (ERO)
        1. Swap ith and jth row
           1. The new determinant will be equal to –det(A) where A was the old matrix. Therefore, multiply the final determinant by -1.
        2. Multiply a row by a Scalar
           1. The new determinant will be equal to kdet(A) where A was the old matrix and k the scalar. Therefore, multiply by 1/k.
        3. Replace with Self and Scalar of Another Row
           1. No Change!
  2. The Determinant of a Linear Transformation
     1. For a linear transformation from V to V where V is a finite-dimensional linear space, then if B is a basis of V and B is the B –matrix of T, then we define det (T) = det (B).
        1. The det (T) will remain unchanged no matter which basis we choose!

1. Eigenvalues and Eigenvectors
   1. *Eigenvector*- for an n x n matrix A, a nonzero vector in Rn Э is a scalar multiple of , or



* + 1. The scalar may be equal to 0



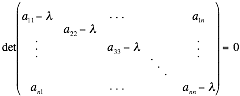
* 1. *Eigenvalue-* the scalar for a particular eigenvector and matrix



* 1. Exponentiation of A
     1. If is an eigenvector of A, then is also an eigenvector of A raised to any power:, , … ,



* 1. Finding the Eigenvalues of a Matrix
     1. *Characteristic Equation-* The relation stating  is true ⇔  is an eigenvalue for the matrix A; also known as the *secular equation*.
        1. This equation is seldom actually written; most people skip straight to:

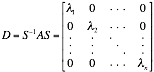


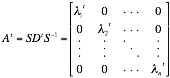
* + - 1. *Characteristic Polynomial-* the polynomial generated by solving for the determinant in the characteristic expression (i.e. finding the determinant of the above), represented by fA().
         1. *Special Case:*  The *2 x 2* Matrix

For a *2 x 2* matrix, the characteristic polynomial is given by:

* + - * 1. If the characteristic polynomial found is incredibly complex, try  or  (common in intro texts)
    1. *Trace-* the sum of the diagonal entries of a square matrix, denoted tr(A)
    2. *Algebraic Multiplicity of an Eigenvalue-* An eigenvalue has algebraic multiplicity k if is a root of multiplicity k of the characteristic polynomial, or rather  for some polynomial  ≠ 0.
    3. Number of Eigenvalues
       1. An *n x n* matrix has--at most--n real eigenvalues, even if they are counted with their algebraic multiplicities.
          1. If n is odd, there exists at least one real eigenvalue
          2. If n is even, there need not exist any real eigenvalues.
    4. Eigenvalues, the Determinant, and the Trace
       1. If an *n x n* matrix A has eigenvalues  listed with their algebraic multiplicities, then
          1. 
          2. 
    5. *Special Case:* Triangular Matrix
       1. The eigenvalues of a Triangular Matrix are its diagonal entries.
    6. *Special Case:* Eigenvalues of Similar Matrices
       1. If matrix A is similar to matrix B (i.e. there exists an invertible S Э B = S-1AS):
          1. A & B have the same characteristic polynomial
          2. rank(A) = rank (B), nullity (A) = nullity (B)
          3. A and B have the same eigenvalues with the same algebraic and geometric multiplicities.

Eigenvectors may be different!

* + - * 1. Matrices A and B have the same determinant and trace.
  1. Finding the Eigenvectors of a Matrix
     1. *Eigenspace-* For a particular eigenvalue  of a matrix A, the kernel of the matrix or 
        1. The eigenvectors with eigenvalue  are the nonzero vectors in the eigenspace.
     2. *Geometric Multiplicity-* the dimension of the eigenspace (the nullity of the matrix ).
        1. If  is an eigenvalue of a square matrix A, then the geometric multiplicity of  must be less than of equal to the algebraic multiplicity of  .
     3. *Eigenbasis-* a basis of Rn consisting of eigenvectors of A for a given *n x n* matrix A.
        1. If an n x n matrix A has n distinct eigenvalues, then there exists an eigenbasis for A.
     4. Eigenbasis and Geometric Multiplicities
        1. By finding the basis of every eigenspace of a given *n x n* matrix A and concatenating them, we can obtain a list of linearly independent eigenvectors (the largest number possible); if the number of elements in this list is equal to n (i.e. the geometric multiplicities sum to n), then we can construct an eigenbasis; otherwise, there doesn’t exist an eigenbasis.
  2. Diagonalization
     1. The process of constructing the matrix of a linear transformation with respect to the eigenbasis of the original matrix of transformation; this always produces a diagonal matrix with the diagonal entries being the transformation’s eigenvalues (recall that eigenvalues are independent of basis; see eigenvalues of similar matrices).
     2. 
     3. *Diagonalizable Matrix-* An n x n matrix A that is similar to some diagonal matrix D.
        1. A matrix A is diagonalizable ⇔ ∃ an eigenbasis for A
        2. If an *n x n* matrix A has n distinct eigenvalues, then A is diagonalizable.
     4. Powers of a Diagonalizable Matrix
        1. To compute the powers At of a diagonalizable matrix (where t is a positive integer), diagonalize A, then raise the diagonal matrix to the t power:



* 1. Complex Eigenvalues
     1. Polar Form
        1. 
     2. De Moivre’s Formula
        1. 
     3. Fundamental Theorem of Algebra
        1. Any polynomial of degree n has, allowing complex numbers and algebraic multiplicity, exactly n (not necessarily distinct) roots.
     4. Finding Complex Eigenvectors
        1. After finding the complex eigenvalues, subtract them along the main diagonal like usual. Afterwards, however, simple take the top row of the resulting matrix, reverse it, multiply by a negative, and voila an eigenvector! This may only work for *2 x 2* matrices.
  2. Discrete Dynamical Systems
     1. Many relations can be represented as a ‘dynamical system’, where the state of the system is given, at any time, by the equation  or equivalently (by exponentiation)  where  represents the basis state and  represents the state at any particular time, t.
     2. This can be further simplified by first finding a basis  of Rn such that  are eigenvectors of the matrix A. Then, using , the exponentiated matrix At can be distributed and the eigenvector properties of  utilized to generate .
     3. The long term properties of a system can be obtained by taking the  and noticing which vectors  approach zero and which approach infinity (in the long run the former will have no effect while the later will attempt to pull the system into an asymptotic approach to themselves (albeit a scaled version of themselves)).
     4. Stability
        1. Dynamical Systems can be divided into two different categories based on their long-term behavior: stable and unstable.
           1. Stable Equilibrium

In the long run, a stable dynamical system asymptotically approaches the zero state () or the original state.

This occurs if the absolute value of all the eigenvalues in are less than (approaches ) or equal to 1 (approaches original or alternates).

* + - * 1. Unstable

 > 1

* + - 1. Polar
         1. This distinction can be transferred into polar coordinates by the following:



* + - * 1. Therefore: 
        2. 
        3. 
    1. Complex Dynamical Systems